

A splitting theorem for equifocal submanifolds with non-flat section

Naoyuki Koike

Abstract

We first prove a certain kind of splitting theorem for an equifocal submanifold with non-flat section in a simply connected symmetric space of compact type, where an equifocal submanifold means a submanifold with parallel focal structure. By using the splitting theorem, we prove that each section of an equifocal submanifold with non-flat section in an irreducible simply connected symmetric space of compact type is isometric to a sphere or a real projective space.

1. Introduction

A properly immersed complete submanifold M in a simply connected symmetric space G/K is called a *submanifold with parallel focal structure* if the following conditions hold:

(PF-i) the restricted normal holonomy group of M is trivial,

(PF-ii) if v is a parallel normal vector field on M such that v_{x_0} is a multiplicity k focal normal of M for some $x_0 \in M$, then v_x is a multiplicity k focal normal of M for all $x \in M$,

(PF-iii) for each $x \in M$, there exists a properly embedded complete connected submanifold through x meeting all parallel submanifolds of M orthogonally.

This notion was introduced by Ewert ([E2]). In [A], [AG] and [AT], this submanifold is simply called an *equifocal submanifold*. In this paper, we also shall use this name and assume that all equifocal submanifolds have trivial normal holonomy group. The submanifold as in (PF-iii) is called a *section of M through x* , which is automatically totally geodesic. Note that Terng-Thorbergsson [TeTh] originally introduced the notion

of an equifocal submanifold under the assumption that the sections is flat. The condition (PF-ii) is equivalent to the following condition:

(PF-ii') for each parallel unit normal vector field v of M , the set of all focal radii along the geodesic γ_{v_x} with $\dot{\gamma}_{v_x}(0) = v_x$ is independent of the choice of $x \in M$.

Note that, under the condition (PF-i), the condition (PF-iii) is equivalent to the following condition:

(PF-iii') M has Lie triple systematic normal bundle (in the sense of [Koi1]).

In fact, (PF-iii) \Rightarrow (PF-iii') is trivial and (PF-iii') \Rightarrow (PF-iii) is shown as follows. If (PF-iii') holds, then it is shown by Proposition 2.2 of [HLO] that $\exp^\perp(T_x^\perp M)$ meets all parallel submanifolds of M orthogonally for each $x \in M$, where \exp^\perp is the normal exponential map of M . Also, it is clear that $\exp^\perp(T_x^\perp M)$ is properly embedded. Thus (PF-iii) follows. An isometric action of a compact Lie group H on a Riemannian manifold is said to be *polar* if there exists a properly embedded complete connected submanifold Σ meeting every principal orbits of the H -action orthogonally. The submanifold Σ is called a *section* of the action. If Σ is flat, then the action is said to be *hyperpolar*. Principal orbits of polar actions are equifocal submanifolds and those of hyperpolar actions are equifocal ones with flat section. Conversely, homogeneous equifocal submanifolds (resp. homogeneous equifocal ones with flat section) in the symmetric spaces are caught as principal orbits of polar (resp. hyperpolar) actions on the spaces.

In 1997, Heintze and Liu [HL] showed that an isoparametric submanifold in a Hilbert space is decomposed into a non-trivial (extrinsic) product of two such submanifolds if and only if the associated Coxeter group is decomposable. In 1998, by using this splitting theorem of Heintze-Liu, Ewert [E1] showed that an equifocal submanifold with flat section in a simply connected symmetric space of compact type is decomposed into a non-trivial (extrinsic) product of two such submanifolds if and only if the associated Coxeter group is decomposable.

In this paper, we first prove the following splitting theorem for an equifocal submanifold with non-flat section in a simply connected symmetric space of compact type.

Theorem A. *Let M be an equifocal submanifold with non-flat section in a simply connected symmetric space G/K of compact type and Σ be a section of M . Then M is decomposed into a non-trivial extrinsic product of two equifocal submanifolds if and only if the restricted*

holonomy group of (the induced metric on) Σ is reducible.

Next we prove the following fact in terms of Theorem A.

Theorem B. *Let M be an equifocal submanifold with non-flat section in an irreducible simply connected symmetric space G/K of compact type. Then each section of M is isometric to a sphere or a real projective space.*

Proof of Theorems A and B

In this section, we shall prove Theorem A. Without loss of generality, we may assume that G is simply connected and K is connected. Let $\pi : G \rightarrow G/K$ be the natural projection and $\phi : H^0([0, 1], \mathfrak{g}) \rightarrow G$ be the parallel transport map for G , where \mathfrak{g} is the Lie algebra of G and $H^0([0, 1], \mathfrak{g})$ is the space of all L^2 -integrable paths having $[0, 1]$ as the domain. Let $M^* := \pi^{-1}(M)$ and $\widetilde{M} := (\pi \circ \phi)^{-1}(M)$. Since G is simply connected and K is connected, M^* and \widetilde{M} are connected. Denote by A (resp. \widetilde{A}) the shape tensor of M (resp. \widetilde{M}) and by ∇^\perp (resp. $\widetilde{\nabla}^\perp$) the normal connection of M (resp. \widetilde{M}). Let Σ_x be the section of M through x ($x \in M$). Assume that the restricted holonomy group of Σ_x is reducible. Fix $x_0 \in M$. We have the non-trivial orthogonal decomposition $T_x \Sigma_x = W_1 \oplus W_2$, which is invariant with respect to the restricted holonomy group of Σ_{x_0} at x_0 . Since M has trivial normal holonomy group, there exists the ∇^\perp -parallel subbundle D_i^N of the normal bundle $T^\perp M$ of M with $(D_i^N)_{x_0} = W_i$ ($i = 1, 2$). For each $x \in M$, it is easy to show that there exists an isometry f of a neighborhood of x_0 in Σ_{x_0} onto a neighborhood of x in Σ_x such that f_{*x_0} coincides with the parallel translation (with respect to ∇^\perp) along any curve in M from x_0 to x . From this fact, it follows that, for each $x \in M$, the orthogonal decomposition $T_x \Sigma_x = (D_1^N)_x \oplus (D_2^N)_x$ is invariant with respect to the restricted holonomy group of Σ_x at x . Let \widetilde{D}_i^N ($i = 1, 2$) be the subbundles of the normal bundle $T^\perp \widetilde{M}$ of \widetilde{M} with $(\pi \circ \phi)_* u((\widetilde{D}_i^N)_u) = (D_i^N)_{(\pi \circ \phi)(u)}$ ($u \in \widetilde{M}$) and D_i^{N*} ($i = 1, 2$) be those of $T^\perp(M^*)$ with $\pi_*((D_i^{N*})_g) = (D_i^N)_{\pi(g)}$ ($g \in G$). According to Lemma 1A.4 of [PoTh1], the focal set of (M, x) consists of finitely many totally geodesic hypersurfaces in Σ_x . Denote by \mathfrak{L}_x the set of all focal hypersurfaces of (M, x) . Let $\psi_x : \widehat{\Sigma}_x \rightarrow \Sigma_x$ be the universal covering of Σ_x . According to the de Rham's decomposition theorem, $\widehat{\Sigma}_x$ is isometric to the (non-trivial) Riemannian product $\widehat{\Sigma}_x^1 \times \widehat{\Sigma}_x^2$, where $\widehat{\Sigma}_x^i$ ($i = 1, 2$) is the complete totally geodesic submanifold of $\widehat{\Sigma}_x$ through

$\hat{x} \in \psi_x^{-1}(x)$ such that $(\psi_x)_{*\hat{x}}(T_{\hat{x}}\hat{\Sigma}_x^i) = (D_i^N)_x$. By retaking the decomposition $T_{x_0}^\perp M = W_1 \oplus W_2$ if necessary, we may assume that $\hat{\Sigma}_x^1$ has no Euclidean part in the de Rham's decomposition for each $x \in M$. Let $\hat{\mathfrak{L}}_x := \{\psi_x^{-1}(L) \mid L \in \mathfrak{L}_x\}$. According to Corollary 3.6 of [Kol2], elements of $\hat{\mathfrak{L}}_x$ are either $L_1 \times \hat{\Sigma}_x^2$ -type (L_1 : a totally geodesic hypersurface of $\hat{\Sigma}_x^1$) or $\hat{\Sigma}_x^1 \times L_2$ -type (L_2 : a totally geodesic hypersurface of $\hat{\Sigma}_x^2$), where we need the fact that $\hat{\Sigma}_x^1$ has no Euclidean part. Denote by $\hat{\mathfrak{L}}_x^1$ (resp. $\hat{\mathfrak{L}}_x^2$) the set of all elements of $\hat{\mathfrak{L}}_x$ of $L_1 \times \hat{\Sigma}_x^2$ -type (resp. of $\hat{\Sigma}_x^1 \times L_2$ -type) and set $\mathfrak{L}_x^i := \{L \in \mathfrak{L}_x \mid \psi_x^{-1}(L) \in \hat{\mathfrak{L}}_x^i\}$ ($i = 1, 2$). Let $V' := \text{Span}(\bigcup_{u \in \widetilde{M}} T_u^\perp \widetilde{M})$, $V_i := \text{Span}(\bigcup_{u \in \widetilde{M}} (\widetilde{D}_i^N)_u)$ ($i = 1, 2$) and $V_0 := (V')^\perp$.

Also, let $(\widetilde{D}_0^T)_u := \bigcap_{v \in T_u^\perp \widetilde{M}} \text{Ker } \widetilde{A}_v$, $(\widetilde{D}_1^T)_u := \left(\bigcap_{v \in (\widetilde{D}_2^N)_u} \text{Ker } \widetilde{A}_v \right) \ominus (\widetilde{D}_0^T)_u$

and $(\widetilde{D}_2^T)_u := \left(\bigcap_{v \in (\widetilde{D}_1^N)_u} \text{Ker } \widetilde{A}_v \right) \ominus (\widetilde{D}_0^T)_u$, where $u \in \widetilde{M}$. Without loss

of generality, we may assume that \widetilde{M} includes the zero element $\hat{0}$ of $H^0([0, 1], \mathfrak{g})$, where we note that $\hat{0}$ is the constant path at the zero element 0 of \mathfrak{g} . Let $\widetilde{M}' := \widetilde{M} \cap V'$. First we prepare the following fact.

Proposition 2.1. *We have $\widetilde{M} = \widetilde{M}' \times V_0 \subset V' \times V_0 = H^0([0, 1], \mathfrak{g})$.*

Proof. First we shall show $V_0 \subset (\widetilde{D}_0^T)_u$ for each $u \in \widetilde{M}$, where we regard $(\widetilde{D}_0^T)_u \subset T_u H^0([0, 1], \mathfrak{g})$ as a subspace of $H^0([0, 1], \mathfrak{g})$ under the identification of $T_u H^0([0, 1], \mathfrak{g})$ with $H^0([0, 1], \mathfrak{g})$. From the definition of V_0 , we have $V_0 \subset T_u \widetilde{M}$ for each $u \in \widetilde{M}$. Let $(\widetilde{D}_0^T)_u^\perp$ be the orthogonal complement of $(\widetilde{D}_0^T)_u$ in $T_u \widetilde{M}$. Clearly we have $(\widetilde{D}_0^T)_u^\perp =$

$\sum_{v \in T_u^\perp \widetilde{M}} \left(\bigoplus_{\lambda \in \text{Spec } \widetilde{A}_v \setminus \{0\}} \text{Ker}(\widetilde{A}_v - \lambda \text{id}) \right)$, where $\text{Spec } \widetilde{A}_v$ is the spectrum of

\widetilde{A}_v . Let $X \in \text{Ker}(\widetilde{A}_v - \lambda \text{id})$ ($v \in T_u^\perp \widetilde{M}$, $\lambda \in \text{Spec } \widetilde{A}_v \setminus \{0\}$). Let J_X be the strongly Jacobi field along the normal geodesic γ_v with $\gamma_v'(0) = v$ satisfying $J_X(0) = X$ (hence $J_X'(0) = -A_v X$). Let $\alpha : (-\varepsilon, \varepsilon) \rightarrow M$ be a curve in M with $\alpha'(0) = X$ and \widetilde{v} be the parallel normal vector field along α with $\widetilde{v}_0 = v$. Define a map $\delta : (-\varepsilon, \varepsilon) \times [0, \infty) \rightarrow H^0([0, 1], \mathfrak{g})$ by $\delta(t, s) := \gamma_{\widetilde{v}_t}(s)$, where $\gamma_{\widetilde{v}_t}$ is the normal geodesic in $H^0([0, 1], \mathfrak{g})$ with $\gamma_{\widetilde{v}_t}'(0) = \widetilde{v}_t$. Then we have $\delta_*(\frac{\partial}{\partial t}|_{t=0}) = J_X$. Since $\delta(t, 0) - \delta(t, \frac{1}{\lambda}) \in T_{\alpha(t)}^\perp \widetilde{M} \subset V'$ for each $t \in (-\varepsilon, \varepsilon)$, we have $\delta_*(\frac{\partial}{\partial t}|_{t=s=0}) - \delta_*(\frac{\partial}{\partial t}|_{t=0, s=\frac{1}{\lambda}}) \in V'$. On the other hand, we have $\delta_*(\frac{\partial}{\partial t}|_{t=s=0}) = X$ and $\delta_*(\frac{\partial}{\partial t}|_{t=0, s=\frac{1}{\lambda}}) = 0$. Hence we have $X \in V'$. From the arbitrariness of X , it follows that

$\text{Ker}(\tilde{A}_v - \lambda \text{id}) \subset V'$. Furthermore, it follows from the arbitrarinesses of λ and v that $(\tilde{D}_0^T)_u^\perp \subset V'$, that is, $V_0 \subset (\tilde{D}_0^T)_u$. Since $V_0 \subset (\tilde{D}_0^T)_u \subset T_u \tilde{M}$ for any $u \in \tilde{M}$, we have $\tilde{M} = \bigcup_{u \in \tilde{M}'} (u + V_0) = \tilde{M}' \times V_0 \subset V' \times V_0$. q.e.d.

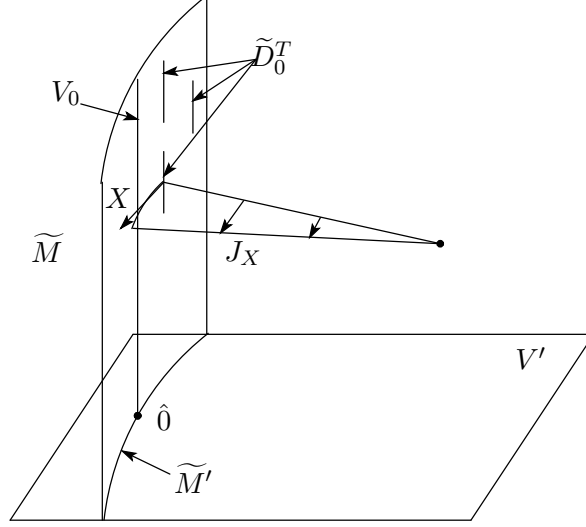


Fig. 1.

Define distributions D_0^T , D_1^T and D_2^T on M by

$$\begin{aligned} (D_0^T)_x &:= \left(\bigcap_{v \in T_x^\perp M} \text{Ker } A_v \right) \cap g_* \left(\mathfrak{c}_{g_*^{-1}T_x M} (g_*^{-1}T_x^\perp M) \right), \\ (D_1^T)_x &:= \left(\left(\bigcap_{v \in (D_2^N)_x} \text{Ker } A_v \right) \cap g_* \left(\mathfrak{c}_{g_*^{-1}T_x M} (g_*^{-1}(D_2^N)_x) \right) \right) \ominus (D_0^T)_x, \\ (D_2^T)_x &:= \left(\left(\bigcap_{v \in (D_1^N)_x} \text{Ker } A_v \right) \cap g_* \left(\mathfrak{c}_{g_*^{-1}T_x M} (g_*^{-1}(D_1^N)_x) \right) \right) \ominus (D_0^T)_x, \end{aligned}$$

for each $x = gK \in M$, where $\mathfrak{c}_*(\sharp)$ is the centralizer of \sharp in $*$. Take an arbitrary $v \in T_{eK}^\perp M$. Set $\mathfrak{p} := T_{eK}(G/K)$. Denote by $\text{Spec } R(v)$ the spectrum of $R(v) := R(\cdot, v)v$, where R is the curvature tensor of G/K . For $\mu \in \text{Spec } R(v)$, we set $\mathfrak{p}_\mu^v := \text{Ker}(R(v) - \mu \text{id})$, $\mathfrak{f}_\mu^v := \text{ad}(v)\mathfrak{p}_\mu^v$ ($\mu \in \text{Spec } R(v) \setminus \{0\}$) and $\mathfrak{f}_0^v := (\text{Ker } \text{ad}(v)) \cap \mathfrak{p}$. Note that

$$(2.2) \quad T_{eK} M = \mathfrak{p}_0^v \cap T_{eK} M + \sum_{\mu \in \text{Spec } R(v) \setminus \{0\}} (\mathfrak{p}_\mu^v \cap T_{eK} M)$$

and

$$(2.3) \quad T_{eK}^\perp M = \mathfrak{p}_0^v \cap T_{eK}^\perp M + \sum_{\mu \in \text{Spec } R(v) \setminus \{0\}} (\mathfrak{p}_\mu^v \cap T_{eK}^\perp M)$$

because M is equifocal and hence it has Lie triple systematic normal bundle. Denote by \mathfrak{f} the Lie algebra of K . For each $X \in \mathfrak{g}$, we define loop vectors $l_{X,k}^{co}$ and $l_{x,k}^{si}$ ($k \in \mathbf{N}$) by $l_{X,k}^{co}(t) := X \cos(2k\pi t)$ and $l_{X,k}^{si}(t) := X \sin(2k\pi t)$. For $X \in \mathfrak{p}_\mu^v$ ($\mu \in \text{Spec } R(v) \setminus \{0\}$), we set $X_{\mathfrak{f}} := \frac{1}{\sqrt{\mu}} \text{ad}(v)(X)$. For $X \in \mathfrak{p}_\mu^v$, $Y \in \mathfrak{p}_0^v \oplus \mathfrak{f}_0^v$ and $k \in \mathbf{Z}$, we define loop vectors $l_{v,X,k}^i$, $l_{v,X_{\mathfrak{f}},k}^i$ and $l_{v,Y,k}^i \in H^0([0,1], \mathfrak{g})$ ($i = 1, 2$) by

$$\begin{aligned} l_{v,X,k}^1(t) &= l_{v,X_{\mathfrak{f}},k}^1(t) = l_{X,k}^{co}(t) - l_{X_{\mathfrak{f}},k}^{si}(t), \\ l_{v,X,k}^2(t) &= l_{v,X_{\mathfrak{f}},k}^2(t) = l_{X,k}^{si}(t) + l_{X_{\mathfrak{f}},k}^{co}(t). \end{aligned}$$

For a general $Z \in \mathfrak{g}$, we define loop vectors $l_{v,Z,k}^i \in H^0([0,1], \mathfrak{g})$ ($i = 1, 2, k \in \mathbf{Z}$) by

$$l_{v,Z,k}^i := l_{v,Z_0,k}^i + \sum_{\mu \in \text{Spec } R(v) \setminus \{0\}} \left(l_{v,Z_{\mathfrak{p},\mu},k}^i + l_{v,Z_{\mathfrak{f},\mu},k}^i \right),$$

where $Z = Z_0 + \sum_{\mu \in \text{Spec } R(v) \setminus \{0\}} (Z_{\mathfrak{p},\mu} + Z_{\mathfrak{f},\mu})$ ($Z_0 \in \mathfrak{p}_0^v$, $Z_{\mathfrak{p},\mu} \in \mathfrak{p}_\mu^v$, $Z_{\mathfrak{f},\mu} \in \mathfrak{f}_\mu^v$).

Denote by $\hat{*}$ the constant path at $* \in \mathfrak{g}$. Note that $\hat{*}$ is the horizontal lift of $*$ ($\in \mathfrak{g} = T_e G$) to $\hat{0}$. Then, according to Propositions 3.1 and 3.2 of [Koi2] and those proofs, we have the following relations.

Lemma 2.2. *Let $X \in T_{eK}M \cap \mathfrak{p}_\mu^v$. Then we have*

$$\begin{aligned} \tilde{A}_{\hat{v}} l_{v,X,k}^1 &= \frac{\sqrt{\mu}}{2k\pi} (\hat{X} - l_{v,X,k}^1), \\ \tilde{A}_{\hat{v}} l_{v,X,k}^2 &= \frac{\sqrt{\mu}}{2k\pi} (\hat{X}_{\mathfrak{f}} - l_{v,X,k}^2), \\ \tilde{A}_{\hat{v}} \hat{X} &= \widehat{A_v X} - \frac{\sqrt{\mu}}{2} \hat{X}_{\mathfrak{f}} + \frac{\sqrt{\mu}}{2\pi} \sum_{k \in \mathbf{Z} \setminus \{0\}} \frac{1}{k} l_{v,X,k}^1, \\ \tilde{A}_{\hat{v}} \hat{X}_{\mathfrak{f}} &= -\frac{\sqrt{\mu}}{2} \hat{X} + \frac{\sqrt{\mu}}{2\pi} \sum_{k \in \mathbf{Z} \setminus \{0\}} \frac{1}{k} l_{v,X,k}^2 \end{aligned}$$

and

$$\tilde{\nabla}_{l_{v,X,k}^1}^\perp \tilde{v}^L = \tilde{\nabla}_{l_{v,X,k}^2}^\perp \tilde{v}^L = \tilde{\nabla}_{\hat{X}}^\perp \tilde{v}^L = \tilde{\nabla}_{\hat{X}_{\mathfrak{f}}}^\perp \tilde{v}^L = 0,$$

where $k \in \mathbf{Z} \setminus \{0\}$ and \tilde{v}^L is the horizontal lift of a parallel normal vector field \tilde{v} with $\tilde{v}_0 = v$ along an arbitrary curve α in M with $\dot{\alpha}(0) = X$.

Lemma 2.3. Let $w \in T_{eK}^\perp M \cap \mathfrak{p}_\alpha^v$. Then we have

$$\begin{aligned}\tilde{A}_{\hat{v}} l_{v,w,k}^1 &= -\frac{\sqrt{\mu}}{2k\pi} l_{v,w,k}^1, \\ \tilde{A}_{\hat{v}} l_{v,w,k}^2 &= \frac{\sqrt{\mu}}{2k\pi} (\hat{w}_{\mathfrak{f}} - l_{v,w,k}^2), \\ \tilde{A}_{\hat{v}} \hat{w}_{\mathfrak{f}} &= \frac{\sqrt{\mu}}{2\pi} \sum_{k \in \mathbf{Z} \setminus \{0\}} \frac{1}{k} l_{v,w,k}^2,\end{aligned}$$

and

$$\tilde{\nabla}_{l_{v,w,k}^1}^\perp \tilde{v}^L = -\frac{\sqrt{\mu}}{2k\pi} \hat{w}, \quad \tilde{\nabla}_{l_{v,w,k}^2}^\perp \tilde{v}^L = 0, \quad \tilde{\nabla}_{\hat{w}_{\mathfrak{f}}}^\perp \tilde{v}^L = \frac{\sqrt{\mu}}{2} \hat{w},$$

where $k \in \mathbf{Z} \setminus \{0\}$ and \tilde{v}^L is as in Lemma 2.2.

Lemma 2.4. Let $X \in \mathfrak{p}_0^v$ and $Y \in \mathfrak{f}_0^v$. Then we have

$$\begin{aligned}\tilde{A}_{\hat{v}} l_{X,k}^{co} &= \tilde{A}_{\hat{v}} l_{X,k}^{si} = \tilde{A}_{\hat{v}} l_{Y,k}^{co} = \tilde{A}_{\hat{v}} l_{Y,k}^{si} = \tilde{A}_{\hat{v}} \hat{Y} = 0, \\ \tilde{\nabla}_{l_{X,k}^{co}}^\perp \tilde{v}^L &= \tilde{\nabla}_{l_{X,k}^{si}}^\perp \tilde{v}^L = \tilde{\nabla}_{l_{Y,k}^{co}}^\perp \tilde{v}^L = \tilde{\nabla}_{l_{Y,k}^{si}}^\perp \tilde{v}^L = \tilde{\nabla}_{\hat{Y}}^\perp \tilde{v}^L = 0\end{aligned}$$

and

$$\tilde{A}_{\hat{v}} \hat{X} = \widehat{A_v X}, \quad \tilde{\nabla}_{\hat{X}}^\perp \tilde{v}^L = 0 \quad (\text{when } X \in \mathfrak{p}_0^v \cap T_{eK} M),$$

where $i = 1, 2$, $k \in \mathbf{N}$ and \tilde{v}^L is as in Lemma 2.2.

From Lemmas 2.2 ~ 2.4, we can show the following relations.

Lemma 2.5. At $\hat{0} \in \widetilde{M}$, $(\tilde{D}_0^T)_{\hat{0}}$ is equal to

$$\begin{aligned}\text{Span}\{\hat{X} \mid X \in (D_0^T)_{eK}\} &\oplus \text{Span}\{\hat{\eta} \mid \eta \in \mathfrak{c}_{\mathfrak{f}}(T_{eK}^\perp M)\} \\ &\oplus \text{Span}\{l_{Z,k}^{co} \mid Z \in \mathfrak{c}_{\mathfrak{g}}(T_{eK}^\perp M), k \in \mathbf{N} \setminus \{0\}\} \\ &\oplus \text{Span}\{l_{Z,k}^{si} \mid Z \in \mathfrak{c}_{\mathfrak{g}}(T_{eK}^\perp M), k \in \mathbf{N} \setminus \{0\}\}\end{aligned}$$

and $(\tilde{D}_{j_1}^T)_{\hat{0}}$ is equal to

$$\begin{aligned}\text{Span}\{\hat{X} \mid X \in (D_{j_1}^T)_{eK}\} &\oplus \text{Span}\{\hat{\eta} \mid \eta \in \mathfrak{c}_{\mathfrak{f}}((D_{j_2}^N)_{eK}) \ominus \mathfrak{c}_{\mathfrak{f}}(T_{eK}^\perp M)\} \\ &\oplus \text{Span}\{l_{Z,k}^{co} \mid Z \in \mathfrak{c}_{\mathfrak{g}}((D_{j_2}^N)_{eK}) \ominus \mathfrak{c}_{\mathfrak{g}}(T_{eK}^\perp M), k \in \mathbf{N} \setminus \{0\}\} \\ &\oplus \text{Span}\{l_{Z,k}^{si} \mid Z \in \mathfrak{c}_{\mathfrak{g}}((D_{j_2}^N)_{eK}) \ominus \mathfrak{c}_{\mathfrak{g}}(T_{eK}^\perp M), k \in \mathbf{N} \setminus \{0\}\},\end{aligned}$$

where $(j_1, j_2) = (1, 2)$ or $(2, 1)$.

Proof. According to Lemmas 2.2 ~ 2.4, we have

$$\begin{aligned}\text{Ker } \tilde{A}_{\hat{v}} &= \text{Span}\{\hat{X} \mid X \in \text{Ker } A_v \cap \mathfrak{p}_0^v\} \oplus \text{Span}\{\hat{\eta} \mid \eta \in \mathfrak{f}_0^v\} \\ &\oplus \text{Span}\{l_{Z,k}^{co} \mid Z \in \text{Ker } \text{ad}(v), k \in \mathbf{N} \setminus \{0\}\} \\ &\oplus \text{Span}\{l_{Z,k}^{si} \mid Z \in \text{Ker } \text{ad}(v), k \in \mathbf{N} \setminus \{0\}\}.\end{aligned}$$

Hence we have the desired relations.

q.e.d.

From Lemmas 2.2 ~ 2.4, we have the following lemma.

Lemma 2.6. *Assume that $v \in D_i^N$. Let \tilde{v}^L be as in Lemma 2.2. Then the statements (i) and (ii) hold.*

- (i) *For each $X \in T\tilde{M}$, we have $\tilde{\nabla}_X^\perp \tilde{v}^L \in \tilde{D}_i^N$.*
- (ii) *For each $Y \in \tilde{D}_j^T \oplus \tilde{D}_0^T$ ($j \neq i$), we have $\tilde{\nabla}_Y \tilde{v}^L = 0$.*

Proof. Without loss of generality, we may assume that the base point of X is $\hat{0}$. First we shall show the statement (i). According to (2.2), (2.3) and Lemmas 2.2 ~ 2.4, we have only to show $\tilde{\nabla}_X^\perp \tilde{v}^L \in \tilde{D}_i^N$ in case of $X = l_{v,w,k}^1$ or \hat{w}_i ($w \in T_{eK}^\perp M \cap \mathfrak{p}_\mu^v$). Suppose that $w \in D_j^N$ ($j \neq i$). Then we have $\mu = 0$ because the sectional curvature of $\text{Span}\{v, w\}$ is equal to 0. This contradicts $\mu \neq 0$. Hence we have $w \in D_i^N$. Therefore, it follows from Lemma 2.3 that $\tilde{\nabla}_X^\perp \tilde{v}^L \in \text{Span}\{\hat{w}\} \subset \tilde{D}_i^N$. Thus the statement (i) is shown. Next we shall show the statement (ii). From (i), we have $\tilde{\nabla}_Y^\perp \tilde{v}^L = 0$. Also, from the definitions of \tilde{D}_j^T and \tilde{D}_0^T , we have $\tilde{A}_\hat{v} Y = 0$. Hence, we obtain $\tilde{\nabla}_Y \tilde{v}^L = 0$. q.e.d.

By using (ii) of Lemma 2.6, we prove the following lemma.

Lemma 2.7. *For each $u \in \tilde{M}$, the tangent space $T_u \tilde{M}$ is orthogonally decomposed as $T_u \tilde{M} = (\tilde{D}_1^T)_u \oplus (\tilde{D}_2^T)_u \oplus (\tilde{D}_0^T)_u$.*

Proof. Take unit vectors v_i belonging to $(\tilde{D}_i^N)_u$ ($i = 1, 2$). According to (i) of Lemma 2.6, we have $\tilde{R}^\perp(X, Y)v_1 \in (\tilde{D}_1^N)_u$ for any $X, Y \in T_u \tilde{M}$, where \tilde{R}^\perp is the curvature tensor of the normal connection of \tilde{M} . Hence, it follows from the Ricci equation that $[\tilde{A}_{v_1}, \tilde{A}_{v_2}] = 0$. Therefore, we have (2.1)

$$T_u \tilde{M} = \bigoplus_{\lambda \in \text{Spec } \tilde{A}_{v_1}} \bigoplus_{\mu \in \text{Spec } \tilde{A}_{v_2}} \left(\text{Ker}(\tilde{A}_{v_1} - \lambda \text{id}) \cap \text{Ker}(\tilde{A}_{v_2} - \mu \text{id}) \right),$$

where $\text{Spec } \tilde{A}_{v_i}$ ($i = 1, 2$) is the spectrum of \tilde{A}_{v_i} . Set $\tilde{\mathfrak{L}}_u^i := \{(\pi \circ \phi)|_{T_u^\perp \tilde{M}}\}^{-1}(L) \mid L \in \mathfrak{L}_{\pi(u)}^i\}$ ($i = 1, 2$). The family $\tilde{\mathfrak{L}}_u^1 \cup \tilde{\mathfrak{L}}_u^2$ gives the family of all focal hypersurfaces of \tilde{M} at u . Let $\lambda \in \text{Spec } \tilde{A}_{v_1} \setminus \{0\}$ and $\mu \in \text{Spec } \tilde{A}_{v_2} \setminus \{0\}$. We shall show $\text{Ker}(\tilde{A}_{v_1} - \lambda \text{id}) \cap \text{Ker}(\tilde{A}_{v_2} - \mu \text{id}) = \{0\}$. Suppose that $\text{Ker}(\tilde{A}_{v_1} - \lambda \text{id}) \cap \text{Ker}(\tilde{A}_{v_2} - \mu \text{id}) \neq \{0\}$. Take $X (\neq 0) \in \text{Ker}(\tilde{A}_{v_1} - \lambda \text{id}) \cap \text{Ker}(\tilde{A}_{v_2} - \mu \text{id})$. The point $u + \frac{1}{\lambda}v_1$ and $u + \frac{1}{\mu}v_2$ are focal points along the normal geodesics γ_{v_1} and γ_{v_2} , respec-

tively. Hence there exist $L_1 \in \tilde{\mathfrak{L}}_u^1$ with $u + \frac{1}{\lambda}v_1 \in L_1$ and $L_2 \in \tilde{\mathfrak{L}}_u^2$ with $u + \frac{1}{\mu}v_2 \in L_2$. Let $w_\theta := \cos \theta \cdot v_1 + \frac{\lambda}{\mu} \sin \theta \cdot v_2$ ($0 \leq \theta \leq \frac{\pi}{2}$). Since $A_{w_\theta}X = \lambda(\sin \theta + \cos \theta)X$, the point $u + \frac{1}{\lambda(\sin \theta + \cos \theta)}w_\theta$ is a focal point along γ_{w_θ} for each $\theta \in [0, \frac{\pi}{2}]$. Define a curve $c : [0, \frac{\pi}{2}] \rightarrow H^0([0, 1], \mathfrak{g})$ by $c(\theta) := u + \frac{1}{\lambda(\sin \theta + \cos \theta)}w_\theta$ ($\theta \in I$), which is smooth and regular. For each $\theta \in [0, \frac{\pi}{2}]$, we have $c(\theta) \in \bigcup_{L \in \tilde{\mathfrak{L}}_u^1 \cup \tilde{\mathfrak{L}}_u^2} (L \cap \text{Span}\{v_1, v_2\})$.

For simplicity, we set $F := \bigcup_{L \in \tilde{\mathfrak{L}}_u^1 \cup \tilde{\mathfrak{L}}_u^2} (L \cap \text{Span}\{v_1, v_2\})$. Since F is a family of affine lines in $\text{Span}\{v_1, v_2\}$ which are parallel to $\text{Span}\{v_1\}$ or $\text{Span}\{v_2\}$ and c is a regular curve in F , c lies in the only affine line belonging to F . It is clear that the affine lines $L_1 \cap \text{Span}\{v_1, v_2\}$ and $L_2 \cap \text{Span}\{v_1, v_2\}$ are mutually distinct. These facts contradict $c(0) \in L_1$ and $c(\frac{\pi}{2}) \in L_2$ (see Fig. 2). Therefore we have $\text{Ker}(\tilde{A}_{v_1} - \lambda \text{id}) \cap \text{Ker}(\tilde{A}_{v_2} - \mu \text{id}) = \{0\}$. This fact together with (2.1) deduces $\bigoplus_{\lambda \in \text{Spec } \tilde{A}_{v_1} \setminus \{0\}} \text{Ker}(\tilde{A}_{v_1} - \lambda \text{id}) \subset \text{Ker } \tilde{A}_{v_2}$. From the arbitrariness of v_2 ,

we have $\bigoplus_{\lambda \in \text{Spec } \tilde{A}_{v_1} \setminus \{0\}} \text{Ker}(\tilde{A}_{v_1} - \lambda \text{id}) \subset (\tilde{D}_0^T)_u \oplus (\tilde{D}_1^T)_u$. That is, the

orthogonal complement $((\tilde{D}_0^T)_u \oplus (\tilde{D}_1^T)_u)^\perp$ of $(\tilde{D}_0^T)_u \oplus (\tilde{D}_1^T)_u$ is contained in $\text{Ker } \tilde{A}_{v_1}$. From the arbitrariness of v_1 , we have $((\tilde{D}_0^T)_u \oplus (\tilde{D}_1^T)_u)^\perp \subset (\tilde{D}_0^T)_u \oplus (\tilde{D}_2^T)_u$, which implies $((\tilde{D}_0^T)_u \oplus (\tilde{D}_1^T)_u)^\perp \subset (\tilde{D}_2^T)_u$. On the other hand, we have $((\tilde{D}_0^T)_u \oplus (\tilde{D}_1^T)_u) \cap (\tilde{D}_2^T)_u = \{0\}$. Hence we have $T_u \tilde{M} = (\tilde{D}_0^T)_u \oplus (\tilde{D}_1^T)_u \oplus (\tilde{D}_2^T)_u$ and $((\tilde{D}_0^T)_u \oplus (\tilde{D}_1^T)_u)^\perp = (\tilde{D}_2^T)_u$. After all we have $T_u \tilde{M} = (\tilde{D}_0^T)_u \oplus (\tilde{D}_1^T)_u \oplus (\tilde{D}_2^T)_u$ (orthogonal direct sum).

q.e.d.

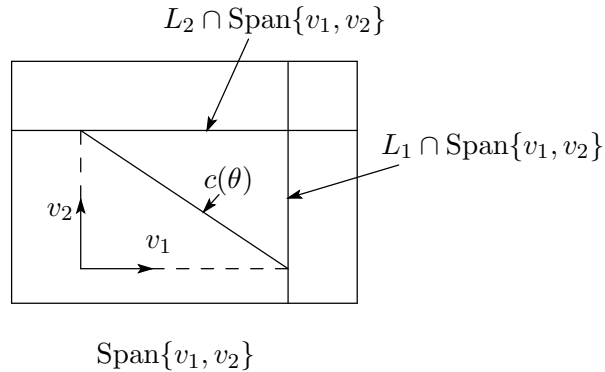


Fig. 2.

Next we prepare the following lemma.

Lemma 2.8. (i) The distributions $\tilde{D}_i^T \oplus \tilde{D}_0^T$ ($i = 1, 2$) are totally geodesic.

(ii) The distributions \tilde{D}_i^T ($i = 1, 2$) are totally geodesic.

Proof. For simplicity, set $\tilde{D}_{i0}^T := \tilde{D}_i^T \oplus \tilde{D}_0^T$ ($i = 1, 2$). Denote by \tilde{h} (resp. \tilde{h}_{10}) the second fundamental form of \tilde{M} (resp. \tilde{D}_{10}^T), by \tilde{A}^{10} the shape tensor of \tilde{D}_{10}^T , by $\tilde{\nabla}$ (resp. $\nabla^{\tilde{M}}$) the Levi-Civita connection of $H^0([0, 1], \mathfrak{g})$ (resp. \tilde{M}) and by ∇^{\perp_2} the normal connection of \tilde{D}_2^T . Also, denote by $\bar{\nabla}$ the connection of the bundle $T^*\tilde{M} \otimes T^*\tilde{M} \otimes T^\perp\tilde{M}$ induced from $\nabla^{\tilde{M}}$ and $\tilde{\nabla}^\perp$. Let $X, Y \in (\tilde{D}_{10}^T)_u$ and $Z \in (\tilde{D}_2^T)_u$. Let \tilde{X} (resp. \tilde{Y}) be a section of \tilde{D}_{10}^T with $\tilde{X}_u = X$ (resp. $\tilde{Y}_u = Y$) and \tilde{Z} be a section of \tilde{D}_2^T with $\tilde{Z}_u = Z$. For any $v_1 \in (\tilde{D}_1^N)_u$, we have $\langle \tilde{h}(Y, Z), v_1 \rangle = \langle \tilde{A}_{v_1} Z, Y \rangle = 0$ because of $(\tilde{D}_2^T)_u \subset \text{Ker } \tilde{A}_{v_1}$. Also, for any $v_2 \in (\tilde{D}_2^N)_u$, we have $\langle \tilde{h}(Y, Z), v_2 \rangle = \langle \tilde{A}_{v_2} Y, Z \rangle = 0$ because of $(\tilde{D}_{10}^T)_u \subset \text{Ker } \tilde{A}_{v_2}$. Hence we have $\tilde{h}(Y, Z) = 0$. From the arbitrarinesses of Y, Z and u , we have $\tilde{h}(\tilde{D}_{10}^T, \tilde{D}_2^T) = 0$. Also, we can show $\tilde{h}(\tilde{D}_{10}^T, \tilde{D}_{10}^T) \subset \tilde{D}_1^N$ and $\tilde{h}(\tilde{D}_2^T, \tilde{D}_2^T) \subset \tilde{D}_2^N$. Let X, Y, Z, \tilde{Y} and \tilde{Z} be as above. It follows from $\tilde{h}(\tilde{D}_{10}^T, \tilde{D}_2^T) = 0$ that

$$\begin{aligned}
(2.2) \quad (\bar{\nabla}_X \tilde{h})(Z, Y) &= \tilde{\nabla}_X^\perp(\tilde{h}(\tilde{Z}, \tilde{Y})) - \tilde{h}(\nabla_X^{\tilde{M}} \tilde{Z}, Y) - \tilde{h}(Z, \nabla_X^{\tilde{M}} \tilde{Y}) \\
&= \tilde{h}(\tilde{A}_Z^{10} X, Y) - \tilde{h}(Z, h_{10}(X, Y)) \\
&\equiv -\tilde{h}(Z, h_{10}(X, Y)) \pmod{(\tilde{D}_1^N)_u}.
\end{aligned}$$

Also, it follows from $\tilde{h}(\tilde{D}_{10}^T, \tilde{D}_{10}^T) \subset \tilde{D}_1^N$ and Lemma 2.6 that

$$\begin{aligned}
(2.3) \quad (\bar{\nabla}_Z \tilde{h})(X, Y) &= \tilde{\nabla}_Z^\perp(\tilde{h}(\tilde{X}, \tilde{Y})) - \tilde{h}(\nabla_Z^{\perp_2} \tilde{X}, Y) - \tilde{h}(X, \nabla_Z^{\perp_2} \tilde{Y}) \\
&\equiv 0 \pmod{(\tilde{D}_1^N)_u}.
\end{aligned}$$

By (2.2), (2.3) and the Codazzi equation, we have $\tilde{h}(Z, h_{10}(X, Y)) \in (\tilde{D}_1^N)_u$. On the other hand, it follows from $\tilde{h}(\tilde{D}_2^T, \tilde{D}_2^T) \subset \tilde{D}_2^N$ that $\tilde{h}(Z, h_{10}(X, Y)) \in (\tilde{D}_2^N)_u$. Hence we have $\tilde{h}(Z, h_{10}(X, Y)) = 0$. According to the proof of Lemma 2.7, we have

$$(\tilde{D}_2^T)_u = \bigoplus_{v_2 \in (\tilde{D}_2^N)_u} \bigoplus_{\mu \in \text{Spec } \tilde{A}_{v_2} \setminus \{0\}} \text{Ker}(\tilde{A}_{v_2} - \mu \text{id}).$$

If $Z \in \text{Ker}(\tilde{A}_{v_2} - \mu \text{id})$ ($\mu \in \text{Spec } \tilde{A}_{v_2} \setminus \{0\}$), then we have

$$\langle \tilde{h}(Z, h_{10}(X, Y)), v_2 \rangle = \langle \tilde{A}_{v_2} Z, h_{10}(X, Y) \rangle = \mu \langle h_{10}(X, Y), Z \rangle = 0,$$

that is, $\langle h_{10}(X, Y), Z \rangle = 0$. From the arbitrariness of $Z \in (\tilde{D}_2^T)_u$, it follows that $h_{10}(X, Y) = 0$. From the arbitrarinesses of X and Y , it follows

that $h_{10} = 0$, that is, \tilde{D}_{10}^T is totally geodesic. Similarly, we can show that \tilde{D}_{20}^T is totally geodesic. By the similar discussion, we can show the statement (ii). q.e.d.

By using Lemmas 2.6~2.8, we show the following fact.

Lemma 2.9. *We have $V' = V_1 \oplus V_2$ (orthogonal direct sum).*

Proof. Clearly we have $V' = V_1 + V_2$. We have only to show $V_1 \perp V_2$. Take arbitrary $u_1, u_2 \in \tilde{M}$ and arbitrary $v_j \in (\tilde{D}_j^N)_{u_i}$ ($(i, j) = (1, 2), (2, 1)$). Define a subset $U(u_1)$ of $H^0([0, 1], \mathfrak{g})$ by $U(u_1) :=$

$\bigcup_{u \in L_{u_1}^{\tilde{D}_{10}^T}} L_u^{\tilde{D}_{20}^T}$, where $L_{u_1}^{\tilde{D}_{10}^T}$ (resp. $L_u^{\tilde{D}_{20}^T}$) is the leaf of \tilde{D}_{10}^T (resp. \tilde{D}_{20}^T)

through u_1 (resp. u). Since \tilde{M} is complete, \tilde{D}_{10}^T is totally geodesic by Lemma 2.8 and \tilde{D}_2^T is the orthogonal complementary distribution of \tilde{D}_{10}^T by Lemma 2.7, \tilde{D}_2^T is an Ehresmann connection for the foliation consisting of integral manifolds of \tilde{D}_{10}^T (see [BH]). Note that the discussions in [BH] are valid in the infinite dimensional case. From the infinite dimensional version of the discussion in [BH], it follows that $U(u_1) = \tilde{M}$. Therefore we have $L_{u_1}^{\tilde{D}_{10}^T} \cap L_{u_2}^{\tilde{D}_{20}^T} \neq \emptyset$. Take $u_3 \in L_{u_1}^{\tilde{D}_{10}^T} \cap L_{u_2}^{\tilde{D}_{20}^T}$ and curves $\alpha_i : [0, 1] \rightarrow L_{u_i}^{\tilde{D}_{10}^T}$ ($i = 1, 2$) with $\alpha_i(0) = u_i$ and $\alpha_i(1) = u_3$. According to (ii) of Lemma 2.6, we have $P_{\alpha_i}^{\tilde{\nabla}}(v_j) \in (\tilde{D}_j^N)_{u_3}$ ($i = 1, 2$), where $P_{\alpha_i}^{\tilde{\nabla}}$ is the parallel translation along α_i with respect to $\tilde{\nabla}$. Hence we obtain $\langle v_1, v_2 \rangle = 0$. Therefore, it follows from the arbitrarinesses of v_1 and v_2 that $V_1 \perp V_2$. q.e.d.

Fix $x_0 \in M$. According to Lemma 2.15 and Proposition 2.16 of [E2], the focal set of (M, x_0) consists of finitely many totally geodesic hypersurfaces in the section Σ_{x_0} through x_0 . Let \mathfrak{L}_{x_0} be the family of all the focal hypersurfaces. The focal hypersurfaces divide Σ_{x_0} into some open cells. Denote by Δ the component containing $0 \in T_{x_0}^\perp M$ of the inverse image by $\exp_{x_0}^\perp$ of the open cell containing x_0 . Define a map $f : M \times \Delta \rightarrow G/K$ by $f(x, v) := \exp_x^\perp(\tilde{v}_x)$ ($(x, v) \in M \times \Delta$), where \tilde{v} is the parallel normal vector field of M with $\tilde{v}_{x_0} = v$. Let $U := f(M \times \Delta)$, which is an open dense subset of G/K consisting of non-focal points of M . For each $v \in \Delta$, denote by M_v the parallel submanifold $\eta_{\tilde{v}}(M)$ of M , where $\eta_{\tilde{v}}$ is the end-point map for \tilde{v} , that is, $\eta_{\tilde{v}}(x) = f(x, v)$ ($x \in M$). Let E_i^N ($i = 1, 2$) be the distribution on U such that $E_i^N|_M = D_i^N$, $E_i^N|_{\Sigma_x}$ is a parallel distribution on Σ_x for

each $x \in M$ and that $E_i^N|_{M_v}$ is a normal parallel subbundle of $T^\perp M_v$ for each $v \in \Delta$. Denote by $(D_i^T)^v$ ($i = 0, 1, 2$) the distributions on M_v corresponding to the distributions D_i^T ($i = 0, 1, 2$) on M . It is shown that $(D_i^T)^v = (\eta_{\tilde{v}})_*(D_i^T)$. For each $i \in \{0, 1, 2\}$, the distributions $(D_i^T)^v$'s ($v \in \Delta$) give a distribution on U . Denote by E_i^T ($i = 0, 1, 2$) this distribution on U . Set $E_i := E_i^T \oplus E_i^N$ and $E_{i0} := E_i^T \oplus E_i^N \oplus E_0^T$ ($i = 1, 2$). Let $\tilde{U} := (\pi \circ \phi)^{-1}(U)$, which is an open dense subset of $H^0([0, 1], \mathfrak{g})$. For each $v \in \Delta$, denote by \tilde{M}_v the submanifold $\eta_{\tilde{v}^L}(\tilde{M})$, where $\eta_{\tilde{v}^L}$ is the end-point map for the horizontal lift \tilde{v}^L of \tilde{v} . Note that $\eta_{\tilde{v}^L}(\tilde{M})$ is not a parallel submanifold of \tilde{M} because \tilde{v}^L is not parallel with respect to the normal connection of \tilde{M} . Let \tilde{E}_i^N ($i = 1, 2$) be the horizontal lift of E_i^N to \tilde{U} . Denote by $(\tilde{D}_i^T)^v$ the distributions on \tilde{M}_v corresponding to the distributions \tilde{D}_i^T ($i = 0, 1, 2$) on \tilde{M} . For each $i \in \{0, 1, 2\}$, the distributions $(\tilde{D}_i^T)^v$'s ($v \in \Delta$) give a distribution on \tilde{U} . Denote by \tilde{E}_i^T ($i = 0, 1, 2$) this distribution. Set $\tilde{E}_i := \tilde{E}_i^T \oplus \tilde{E}_i^N$ and $\tilde{E}_{i0} := \tilde{E}_i^T \oplus \tilde{E}_i^N \oplus \tilde{E}_0^T$ ($i = 1, 2$). By using Lemmas 2.5 and 2.8, we show the following lemma.

Lemma 2.10. (i) *The distributions \tilde{E}_{i0} ($i = 1, 2$) are totally geodesic.*
(ii) *The distributions \tilde{E}_i ($i = 1, 2$) are totally geodesic.*

Proof. For each $X \in \Gamma(TM)$, we define $\overline{X} \in \Gamma(TU)$ by $\overline{X}_{f(x,v)} := (\eta_{\tilde{v}})_*(X_x)$ ($(x, v) \in M \times \Delta$), where $\eta_{\tilde{v}}$ is as above. Also, for each $w \in \Delta$, we define $\overline{w} \in \Gamma(TU)$ by $\overline{w}_{f(x,v)} := P_{\gamma_{\tilde{v}_x}}^{\Sigma_x}(\tilde{w}_x)$ ($(x, v) \in M \times \Delta$), where \tilde{w} is the parallel normal vector field of M with $\tilde{w}_{x_0} = w$ and $P_{\gamma_{\tilde{v}_x}}^{\Sigma_x}$ is the parallel translation along the geodesic $\gamma_{\tilde{v}_x} : [0, 1] \rightarrow \Sigma_x$ with $\gamma'_{\tilde{v}_x}(0) = \tilde{v}_x$ with respect to the Levi-Civita connection of Σ_x . Note that $P_{\gamma_{\tilde{v}_x}}^{\Sigma_x}$ coincides with the parallel translation along $\gamma_{\tilde{v}_x}$ with respect to the Levi-Civita connection of G/K because Σ_x is totally geodesic. Without loss of generality, we may assume $x_0 = eK$. We suffice to show that \tilde{E}_{i0} ($i = 1, 2$) and \tilde{E}_i ($i = 1, 2$) have $\hat{0}$ as a geodesic point. Easily we can show that if $X \in \Gamma(D_i^T)$ (resp. $w \in \Delta \cap (D_j^N)_{eK}$), then $\overline{X} \in \Gamma(E_i^T)$ (resp. $\overline{w} \in \Gamma(E_j^N)$), where $i = 0, 1, 2$ and $j = 1, 2$. We shall show that \tilde{E}_{10} has $\hat{0}$ as a geodesic point. From Lemma 2.5, we have

$$\begin{aligned}
(\tilde{E}_{10})_{\hat{0}} = & \text{Span}\{\hat{X} \mid X \in (D_{10}^T)_{eK}\} \oplus \text{Span}\{\hat{\eta} \mid \eta \in \mathfrak{c}_f((D_2^N)_{eK})\} \\
& \oplus \text{Span}\{l_{Z,k}^{co} \mid Z \in \mathfrak{c}_g((D_2^N)_{eK}), k \in \mathbf{N} \setminus \{0\}\} \\
(2.4) \quad & \oplus \text{Span}\{l_{Z,k}^{si} \mid Z \in \mathfrak{c}_g((D_2^N)_{eK}), k \in \mathbf{N} \setminus \{0\}\} \\
& \oplus \text{Span}\{\hat{w} \mid w \in (D_1^N)_{eK}\}.
\end{aligned}$$

Denote by \tilde{h}_{10} the second fundamental form of \tilde{E}_{10} . First we show

$\tilde{h}_{10}((\tilde{D}_1^N)_{\hat{0}}, (\tilde{D}_1^N)_{\hat{0}}) = 0$. Let $w_1, w_2 \in (D_1^N)_{eK}$. Denote by ∇ , ∇^* and $\tilde{\nabla}$ the Levi-Civita connection of G/K , G and $H^0([0, 1], \mathfrak{g})$. Denote by $(\cdot)^L$ (resp. $(\cdot)^*$) the horizontal lift of (\cdot) to $H^0([0, 1], \mathfrak{g})$ (resp. G). According to Lemmas 2.2 and 2.3 in [Koi2], we have

$$\tilde{\nabla}_{\hat{w}_1} \bar{w}_2^L = (\nabla_{w_1}^* \bar{w}_2^*)_{\hat{0}}^L - t[w_1, w_2] + \frac{1}{2}[w_1, w_2]_{\hat{0}}^L = (\nabla_{w_1} \bar{w}_2)_{\hat{0}}^L - t[w_1, w_2],$$

where $t[w_1, w_2]$ is the H^0 -path in \mathfrak{g} assigning $t[w_1, w_2]$ to each $t \in [0, 1]$. Since E_1^N is totally geodesic, we have $\nabla_{w_1} \bar{w}_2 \in (D_1^N)_{eK}$ and hence $(\nabla_{w_1} \bar{w}_2)_{\hat{0}}^L \in (\tilde{E}_{10})_{\hat{0}}$ by (2.4). Also, we have $[w_1, w_2] \in \mathfrak{c}_{\mathfrak{f}}((D_2^N)_{eK})$ and hence $t[w_1, w_2] \in (\tilde{E}_{10})_{\hat{0}}$ by (2.4). Therefore, we have $\tilde{\nabla}_{\hat{w}_1} \bar{w}_2^L \in (\tilde{E}_{10})_{\hat{0}}$, that is, $\tilde{h}_{10}(\hat{w}_1, \hat{w}_2) = 0$. Thus we have

$$(2.5) \quad \tilde{h}_{10}((\tilde{D}_1^N)_{\hat{0}}, (\tilde{D}_1^N)_{\hat{0}}) = 0.$$

Set $\tilde{E}_{10}^T := \tilde{E}_1^T \oplus \tilde{E}_0^T$. Next we show that $\tilde{h}_{10}((\tilde{E}_{10}^T)_{\hat{0}}, (\tilde{E}_{10}^T)_{\hat{0}}) = 0$. Let $\tilde{X}, \tilde{Y} \in \Gamma(\tilde{E}_{10}^T)$. For each $w \in (D_2^N)_{eK}$, we have

$$\langle \tilde{h}(\tilde{X}_{\hat{0}}, \tilde{Y}_{\hat{0}}), \hat{w} \rangle = \langle \tilde{A}_{\hat{w}} \tilde{X}_{\hat{0}}, \tilde{Y}_{\hat{0}} \rangle = 0$$

from the definition of \tilde{E}_{10}^T . Hence we have $\tilde{h}(\tilde{X}_{\hat{0}}, \tilde{Y}_{\hat{0}}) \in (\tilde{D}_1^N)_{\hat{0}} \subset (\tilde{E}_{10})_{\hat{0}}$. Also, since \tilde{D}_{10}^T is totally geodesic by Lemma 2.8, we have $\nabla_{\tilde{X}_{\hat{0}}}^{\tilde{M}} \tilde{Y} \in (\tilde{D}_{10}^T)_{\hat{0}} \subset (\tilde{E}_{10})_{\hat{0}}$. Therefore, we have $\tilde{h}_{10}(\tilde{X}_{\hat{0}}, \tilde{Y}_{\hat{0}}) = 0$. Thus we have

$$(2.6) \quad \tilde{h}_{10}((\tilde{E}_{10}^T)_{\hat{0}}, (\tilde{E}_{10}^T)_{\hat{0}}) = 0.$$

Next we show $\tilde{h}_{10}((\tilde{E}_{10}^T)_{\hat{0}}, (\tilde{D}_1^N)_{\hat{0}}) = 0$. Let $w \in (D_1^N)_{eK}$. According to (2.4), we suffices to show that $\tilde{h}_{10}(\hat{X}, \hat{w})$ ($X \in (D_{10}^T)_{eK}$), $\tilde{h}_{10}(\hat{\eta}, \hat{w})$ ($\eta \in \mathfrak{c}_{\mathfrak{f}}((D_2^N)_{eK})$), $\tilde{h}_{10}(l_{Z,k}^{co}, \hat{w})$ and $\tilde{h}_{10}(l_{Z,k}^{si}, \hat{w})$ ($Z \in \mathfrak{c}_{\mathfrak{g}}((D_2^N)_{eK})$, $k \in \mathbf{N} \setminus \{0\}$) vanish. According to Lemmas 2.2 and 2.3 in [Koi2], we have $\tilde{\nabla}_{\hat{X}} \bar{w}^L = (\nabla_X \bar{w})_{\hat{0}}^L - t[X, w]$, $\tilde{\nabla}_{\hat{\eta}} \bar{w}^L = -t[\eta, w]$, $\tilde{\nabla}_{l_{Z,k}^{co}} \bar{w}^L = -[\int_0^t l_{Z,k}^{co}(t) dt, w]$ and $\tilde{\nabla}_{l_{Z,k}^{si}} \bar{w}^L = -[\int_0^t l_{Z,k}^{si}(t) dt, w]$. Also, we can show $\nabla_X \bar{w} = -A_w X \in (D_{10}^T)_{eK}$, $[X, w] \in \mathfrak{c}_{\mathfrak{f}}((D_2^N)_{eK})$, $[\eta, w] \in \mathfrak{c}_{\mathfrak{g}}((D_2^N)_{eK}) \cap ((D_{10}^T)_{eK} \oplus (D_1^N)_{eK})$ and $[\int_0^t l_{Z,k}^{co}(t) dt, w], [\int_0^t l_{Z,k}^{si}(t) dt, w] \in \mathfrak{c}_{\mathfrak{g}}((D_2^N)_{eK}) \cap ((D_{10}^T)_{eK} \oplus (D_1^N)_{eK})$ for each fixed $t \in [0, 1]$. Hence it follows from (2.4) that $\tilde{\nabla}_{\hat{X}} \bar{w}^L, \tilde{\nabla}_{\hat{\eta}} \bar{w}^L, \tilde{\nabla}_{l_{Z,k}^{co}} \bar{w}^L$ and $\tilde{\nabla}_{l_{Z,k}^{si}} \bar{w}^L$ belong to $(\tilde{E}_{10})_{\hat{0}}$. That is, we have $\tilde{h}_{10}(\hat{X}, \hat{w}) = \tilde{h}_{10}(\hat{\eta}, \hat{w}) = \tilde{h}_{10}(l_{Z,k}^{co}, \hat{w}) = \tilde{h}_{10}(l_{Z,k}^{si}, \hat{w}) = 0$. Thus we have

$$(2.7) \quad \tilde{h}_{10}((\tilde{E}_{10}^T)_{\hat{0}}, (\tilde{D}_1^N)_{\hat{0}}) = 0.$$

Similarly, we can show $\tilde{h}_{10}((\tilde{D}_1^N)_{\hat{0}}, (\tilde{E}_{10}^T)_{\hat{0}}) = 0$, which together with (2.5) \sim (2.7) and $(\tilde{E}_{10})_{\hat{0}} = (\tilde{E}_{10}^T)_{\hat{0}} \oplus (\tilde{D}_1^N)_{\hat{0}}$ implies that $(\tilde{h}_{10})_{\hat{0}} = 0$, that is, $\hat{0}$ is a geodesic point of \tilde{E}_{10} . This completes the proof of the totally geodesicness of \tilde{E}_{10} . Similarly, we can show that \tilde{E}_{20} and \tilde{E}_i ($i = 1, 2$) are totally geodesic. q.e.d.

Let $\widetilde{M}_i(u) := \widetilde{M} \cap (u + V_i)$ and $(F_i)_u := T_u \widetilde{M}_i(u)$ ($u \in \widetilde{M}$, $i = 1, 2$).

Lemma 2.11. *The correspondence $F_i : u \mapsto (F_i)_u$ ($u \in \widetilde{M}$) gives a totally geodesic distribution on \widetilde{M} having $\widetilde{M}_i(u)$'s ($u \in \widetilde{M}$) as integral manifolds, where $i = 1, 2$.*

Proof. Fix $u_0 \in \widetilde{M}$. From (ii) of Lemma 2.6, it follows that $V_i = \text{Span} \left(\bigcup_{u \in L_{u_0}^{\tilde{D}_i^T}} (\tilde{D}_i^N)_u \right)$, where $L_{u_0}^{\tilde{D}_i^T}$ is the leaf of \tilde{D}_i^T through u_0 . On the

other hand, it follows from Lemma 2.10 that $(\tilde{D}_i^N)_u$'s ($u \in L_{u_0}^{\tilde{D}_i^T}$) are contained in $T_{u_0} L_{u_0}^{\tilde{D}_i^T} \oplus (\tilde{D}_i^N)_{u_0}$. Hence we have $V_i \subset T_{u_0} L_{u_0}^{\tilde{D}_i^T} \oplus (\tilde{D}_i^N)_{u_0}$ and hence $\widetilde{M}_i(u_0) \subset L_{u_0}^{\tilde{D}_i^T}$. It is clear that $\widetilde{M}_i(u_0)$ is totally geodesic in $L_{u_0}^{\tilde{D}_i^T}$. Also, according to Lemma 2.8, $L_{u_0}^{\tilde{D}_i^T}$ is totally geodesic in \widetilde{M} . Hence $\widetilde{M}_i(u_0)$ is totally geodesic in \widetilde{M} . This completes the proof. q.e.d.

By using this lemma, we can show the following fact.

Lemma 2.12. *The submanifold $\widetilde{M}_i(u)$'s ($u \in \widetilde{M}$) are integral manifolds of \tilde{D}_i^T ($i = 1, 2$).*

Proof. Let $\widetilde{M}'(u) := \widetilde{M} \cap (u + V')$ ($u \in \widetilde{M}$). Since $V' = V_1 \oplus V_2$ (orthogonal direct sum) by Lemma 2.9, we have $T_u \widetilde{M}'(u) = (F_1)_u \oplus (F_2)_u$ (orthogonal direct sum) for each $u \in \widetilde{M}$. Also, it follows from Lemma 2.7 that $T_u \widetilde{M}'(u) = (\tilde{D}_1^T)_u \oplus (\tilde{D}_2^T)_u$ (orthogonal direct sum) for each $u \in \widetilde{M}$. On the other hand, it follows from the proof of Lemma 2.11 that $(F_i)_u \subset (\tilde{D}_i^T)_u$ ($u \in \widetilde{M}$, $i = 1, 2$). These facts imply $F_i = \tilde{D}_i^T$ ($i = 1, 2$). Hence the statement of this lemma follows. q.e.d.

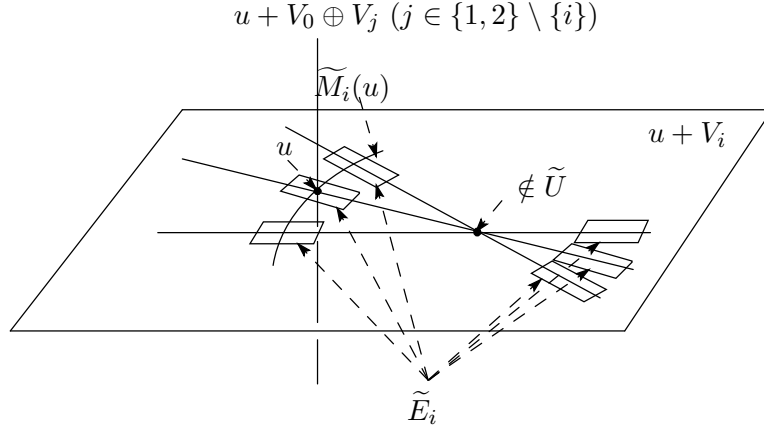


Fig. 3.

By using Lemma 2.11, we can show the following fact.

Lemma 2.13. *For any two points u_1 and u_2 of \widetilde{M}' , $\widetilde{M}_1(u_1)$ intersects with $\widetilde{M}_2(u_2)$.*

Proof. Denote by \mathfrak{F}_1 the foliation on \widetilde{M}' consisting of the integral manifolds of $F_1|_{\widetilde{M}'}$. Since \mathfrak{F}_1 is totally geodesic by Lemma 2.11 and the induced metric on each leaf of \mathfrak{F}_1 is complete, $F_2|_{\widetilde{M}'}$ is an Ehresmann connection for \mathfrak{F}_1 in the sense of Blumenthal-Hebda and hence the statement of this lemma follows (see [BH]). q.e.d.

By using this lemma and imitating the proof of Corollary 3.11 of [HL], we can show the following fact.

Lemma 2.14. *For any $u_0 \in \widetilde{M}_i (= \widetilde{M}_i(\hat{0}))$, the translation map $f_{u_0} : V' \rightarrow V'$ defined by $f_{u_0}(u) := u + u_0$ ($u \in V'$) maps $\widetilde{M}_j (= \widetilde{M}_j(\hat{0}))$ isometrically onto $\widetilde{M}_j(u_0)$, where $(i, j) = (1, 2)$ or $(2, 1)$.*

By using this lemma and imitating the proof of Corollary 3.12 of [HL], we can show the following fact.

Proposition 2.15. *We have $\widetilde{M}' = \widetilde{M}_1 \times \widetilde{M}_2 \subset V_1 \times V_2 = V'$.*

Define ideals \mathfrak{g}' and \mathfrak{g}_i ($i = 1, 2$) by

$$\begin{aligned} \mathfrak{g}' &:= \text{Span} \bigcup_{x^* \in M^*} \{g_{0*}v(x^*)_*^{-1}g_{0*}^{-1} \mid v \in T_{x^*}^\perp M^*, g_0 \in G\}, \\ \mathfrak{g}_i &:= \text{Span} \bigcup_{x^* \in M^*} \{g_{0*}v(x^*)_*^{-1}g_{0*}^{-1} \mid v \in ((D_i^N)_{\pi(x^*)})_{x^*}^L, g_0 \in G\}. \end{aligned}$$

Also, set $\mathfrak{g}_0 := \mathfrak{g} \ominus \mathfrak{g}'$, which is also an ideal of \mathfrak{g} . Let G' and G_i ($i = 0, 1, 2$) be the connected Lie subgroups of G whose Lie algebras are \mathfrak{g}' and \mathfrak{g}_i ($i = 0, 1, 2$), respectively. Since G/K is simply connected, we may assume that G is simply connected. So we have $G = G' \times G_0$ and $G' = G_1 \times G_2$. By imitating the proof of Lemma 5.1 of [Koi4], we can show the following fact.

Lemma 2.16. *We have $V' \subset H^0([0, 1], \mathfrak{g}')$ and $V_i \subset H^0([0, 1], \mathfrak{g}_i)$ ($i = 1, 2$).*

Also, by using Lemma 2.9 and imitating the proof of Lemma 3.7 of [E1], we can show the following fact.

Lemma 2.17. *We have $\mathfrak{g}_1 \perp \mathfrak{g}_2$ and hence $H^0([0, 1], \mathfrak{g}') = H^0([0, 1], \mathfrak{g}_1) \oplus H^0([0, 1], \mathfrak{g}_2)$ (orthogonal direct sum).*

Let $V'_0 := H^0([0, 1], \mathfrak{g}') \ominus V'$ and $V_{i,0} := H^0([0, 1], \mathfrak{g}_i) \ominus V_i$ ($i = 1, 2$). Clearly we have $V'_0 = V_{1,0} \oplus V_{2,0}$. Set $\widetilde{M}'_{H^0} := \widetilde{M} \cap H^0([0, 1], \mathfrak{g}')$ and $\widetilde{M}_{i,H^0} := \widetilde{M} \cap H^0([0, 1], \mathfrak{g}_i)$ ($i = 1, 2$). It follows from Proposition 2.1 that $\widetilde{M}'_{H^0} = \widetilde{M}' \times V'_0$ and $\widetilde{M}_{i,H^0} = \widetilde{M}_i \times V_{i,0}$ ($i = 1, 2$). Furthermore, it follows from Proposition 2.15 that $\widetilde{M} = \widetilde{M}_{1,H^0} \times \widetilde{M}_{2,H^0} \times H^0([0, 1], \mathfrak{g}_0)$. It is clear that the parallel transport map ϕ for G is decomposed as $\phi = \phi_1 \times \phi_2 \times \phi_0$, where ϕ_i ($i = 0, 1, 2$) is the parallel transport map for G_i . Set $M_{i,H^0}^* := \phi_i(\widetilde{M}_{i,H^0})$ ($i = 1, 2$). Clearly we have $M^* = M_{1,H^0}^* \times M_{2,H^0}^* \times G_0 \subset G_1 \times G_2 \times G_0 = G$. Let (\mathfrak{g}, θ) be the orthogonal symmetric Lie algebra of G/K . By imitating the discussion in Section 4 of [E1], we can show the following fact.

Lemma 2.18. *We have $\theta(\mathfrak{g}_i) = \mathfrak{g}_i$ ($i = 0, 1, 2$).*

Let $\mathfrak{f}_i := \text{Fix}(\theta|_{\mathfrak{g}_i})$ and $K_i := \exp_{G_i}(\mathfrak{f}_i)$, where $i = 0, 1, 2$. Since G/K is simply connected, we have $G/K = G_1/K_1 \times G_2/K_2 \times G_0/K_0$. Denote by π_i the natural projection of G_i onto G_i/K_i ($i = 0, 1, 2$). Let $M_{i,H^0} := \pi_i(M_{i,H^0}^*)$ ($i = 1, 2$). Now we prove Theorem A.

Proof of Theorem A. Assume that the holonomy group of the section Σ is reducible. Then, under the above notations, we have $M = M_{1,H^0} \times M_{2,H^0} \times G_0/K_0 \subset G_1/K_1 \times G_2/K_2 \times G_0/K_0 = G/K$. Let $\mathfrak{t} := T_{eK}^\perp M$ and \mathfrak{t}_i ($i = 1, 2$) be the normal space of M_{i,H^0} in G_i/K_i . Since M is equifocal, \mathfrak{t} is a Lie triple system. Hence it follows that \mathfrak{t}_i ($i = 1, 2$)

are Lie triple systems. This fact implies that M_{i,H^0} ($i = 1, 2$) have Lie triple systematic normal bundle. On the other hand, it is clear that M_{i,H^0} ($i = 1, 2$) satisfy the conditions (PF-i) and (PF-ii). Thus M_{i,H^0} ($i = 1, 2$) is equifocal. The converse is trivial. q.e.d.

Next we shall prove Theorem B in terms of Theorem A.

Proof of Theorem B. Let Σ be the section of M through $x_0 = g_0K$ ($\in M$) and $\pi_\Sigma : \widehat{\Sigma} \rightarrow \Sigma$ be the universal covering of Σ . Since G/K is irreducible, it follows from Theorem A that the holonomy group of Σ is irreducible, that is, $\widehat{\Sigma}$ is irreducible. Since Σ is totally geodesic in G/K , it is a symmetric space. Hence $\widehat{\Sigma}$ is an irreducible simply connected symmetric space. On the other hand, according to Lemma 1A.4 of [PoTh1], Σ and hence $\widehat{\Sigma}$ admit a totally geodesic hypersurface. Hence, it follows from the result in [CN] that $\widehat{\Sigma}$ is isometric to a sphere, that is, Σ is isometric to a sphere or a real projective space (of constant curvature). q.e.d.

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Department of Mathematics, Faculty of Science
 Tokyo University of Science, 26 Wakamiya-cho
 Shinjuku-ku, Tokyo 162-8601 Japan
 (koike@ma.kagu.tus.ac.jp)